REMO GARATTINI

On the existence of nontrivial solutions for a nonlinear equation relative to a measure-valued Lagrangian on homogeneous spaces

Abstract. We prove the existence of a non-trivial solution for a nonlinear equation related to a measure-valued Lagrangian. The result is based on a compact embedding theorem of the Lagrangian domain and on the application of the Mountain Pass Theorem joined to a Palais-Smale condition.

1. Introduction and Result

We consider a locally compact separable Hausdorff topological space X endowed with a measure m and a quasidistance d. A quasidistance d on X is a function on $X \times X$ with the usual properties of a metric and a weaker version of the triangle inequality

$$d(x,y) \leq c_T \left(d\left((x,z) + d(z,y)\right)\right), \qquad c_T \geq 1.$$

The set

$$B(x,R) = \{ y \in X : d(x,y) < R \}$$

will be called a quasi-ball. The triple (X,d,m) is assumed to satisfy the following property: for every $R_0 > 0$ there exists a constant $c_0 > 0$, dependent on R_0, such that for $r \leq \frac{R}{2} \leq R \leq R_0$

$$0 < c_0 \left(\frac{r}{R} \right)^{\nu} m\left(B(x,R)\right) \leq m\left(B(x,r)\right) \tag{1.1}$$

for every $x \in X$, where ν is a positive real number independent of r, R, R_0. Such a triple (X,d,m) will be called a homogeneous space of dimension ν. We point out, however, that a given exponent ν occurring in (1.1) should be considered, more precisely, as an upper bound of the "homogeneous dimension", hence we should better call (X,d,m) a homogeneous space of dimension less or equal than ν. Our setting is given by a couple $(X,\mathcal{L})$, "a homogeneous space X with a Lagrangian $\mathcal{L}$", with the following properties

(L1): $\mathcal{L} : \mathcal{C} \longmapsto \mathcal{M}(X)$ is a map which associates with each function u from a given subspace $\mathcal{C}$ of $C(X)$ a measure $\mathcal{L}[u] \in \mathcal{M}^+(X)$, where $C(X)$ denotes the space of all continuous functions on X and $\mathcal{M}^+(X)$ the space of all nonnegative Radon measures on X.

Università degli Studi di Bergamo, Facoltà di Ingegneria, Viale Marconi, 5, 24044 Dalmine (Bergamo) Italy .

E-mail: Garattini@mi.infn.it .

(L2): We assume that there exists $k \geq 1$ such that for a given $p \geq 1$, the following family of Poincaré-like inequalities holds on the metric quasi-balls $B(x,r) \subset\subset X$ [2][3]:

$$\int_{B(x,r)} |u - u_{x,r}|^p \, dm \leq c_P r^p \int_{B(x,kr)} d\mathcal{L}[u], \tag{1.2}$$

where $u_{x,r}$ is the average of u on $B(x,r)$, for every $u \in \mathcal{C}$ and $B(x,r) \subset\subset X$.

(L3): If $u \in \mathcal{C}$ and $g \in C^1(\mathbf{R})$ with g' bounded on $\mathbf{R}$, then $g(u) : x \longmapsto g(u(x))$ belong also to $\mathcal{C}$ and

$$\mathcal{L}[g(u)] = |g'(u)|^p \mathcal{L}[u] \tag{1.3}$$

We are interested in nontrivial solution of the following problem

$$\int_X d\mathcal{L}[u] \, v(x) + \int_X V(x) \, u^p(x) \, v(x) \, m(dx) = \int_X f(u(x)) \, v(x) \, m(dx) \tag{1.4}$$

for every $v \in \mathcal{C} \cap L^p(X, Vm)$ where $u \in \mathcal{C} \cap L^p(X, Vm)$ (Vm is the Radon measure with density V with respect to m). Eq. (1.4) is a generalization of the problem of searching for nontrivial solution for a semilinear equation in the framework of Dirichlet forms as studied in Ref. [4] and in the framework of semilinear equations of the form

$$\triangle u + u^p = 0 \tag{1.5}$$

considered in Ref.[5]. Further developments on semilinear equations for Dirichlet forms can be found in Ref.[6] for problems of the type

$$\int_\Omega \alpha(u,v)(dx) - \lambda \int_\Omega a(x) \, u(x) \, v(x) \, m(dx) = \int_\Omega f(u(x)) \, v(x) \, m(dx),$$

where Ω is an open bounded subset of X, $\alpha(u,v)$ is a uniquely defined signed Radon measure on X, λ is an arbitrary nonvanishing number and $a \in Lip(\bar{\Omega})$ with $a(x) > 0$. To analyze Eq. (1.4), we assume that

$$W = \left\{ u : \int_X d\mathcal{L}[u] + \int_X V u^p m(dx) < +\infty \right\} \tag{1.6}$$

and that

$$\|u\|_W = \left[\int_X d\mathcal{L}[u] + \int_X V u^p m(dx) \right]^{\frac{1}{p}} \tag{1.7}$$

be a norm in W. Moreover let us assume that $V \in C(X, \mathbb{R})$ and

$$V(x) > 0, \qquad \forall x \in X \tag{1.8}$$

$$V(x) \to +\infty, \qquad \text{as} \qquad d(0,x) \to +\infty \tag{1.9}$$

where 0 is an arbitrarily fixed point in X. We assume also that $f(t) \in C(X, \mathbb{R})$ satisfies the following conditions

$$f(0) = 0, \;\; f(t) = o(t), \qquad \text{as } t \to 0 \tag{1.10}$$

$$f(t) = o\left(|t|^{\frac{\nu+p}{\nu-p}}\right), \qquad \text{as } |t| \to +\infty \tag{1.11}$$

if $\nu > p$ or

$$f(t) = o\left(|t|^{\sigma}\right), \qquad \text{as } |t| \to +\infty \tag{1.12}$$

$\sigma > p+1$, if $\nu \leq p$. Finally we assume that

$$0 < \mu F(t) = \mu \int_0^t f(s)\, ds \leq t f(t) \tag{1.13}$$

where $p < \frac{p\nu}{\nu-p}$ if $\nu > p$ or $p < \mu$ if $\nu \leq p$. We observe that from the assumption (1.13) it follows that there exists $m > 0$ such that

$$F(t) \geq m\,|t|^{\mu} \tag{1.14}$$

for $|t| \geq 1$. The result we will prove in the next Section is the following:

Theorem 1. *Let the assumptions* $(1.8), (1.9), (1.10), (1.13)$ *hold together with* (1.11) *if* $\nu > 2$ *or with* (1.12) *if* $\nu = 2$. *Then the problem* (1.4) *has a nontrivial solution.*

Acknowledgments: The Author wishes to thank Marco Biroli for very useful and stimulating discussions on this subject.

2. Preliminary results

We begin the section with a covering Lemma and its Corollary.

Lemma 1. *A ball* $B(x,R)$ *can be covered by a finite number* $n(r,R)$ *of balls* $B(x_i,r)$, $r \leq R$, *such that* $x_i \in B(x,R)$ *and* $B\left(x_i, \frac{r}{2}\right) \cap B\left(x_j, \frac{r}{2}\right) = \emptyset$ *for* $i \neq j$. *Moreover every point of* $B(x,R)$ *is covered by at most* M *balls* $B(x_i,R)$ *where* M *depends on* r.

Proof. The first part of the result follows immediately from assumption (1.1). For the second part we observe that if a point x in $B(x,R)$ is covered by the ball $B(x_i,r)$, then $x_i \in B(x,r)$; so the number M of the balls $B(x_i,r)$, that cover x, is estimated by the greatest number Q of points y_k in $B(x,r)$ with $d(y_{k_1}, y_{k_2}) \geq \frac{r}{2}$ and we observe that, by (1.1), Q is estimated by a number M depending only on r. $\square$

From Lemma 1, we obtain the following

Corollary 1. *The space* X *can be covered by a countable union of balls* $B(x_i,r)$, *such that* $B\left(x_i, \frac{r}{2}\right) \cap B\left(x_j, \frac{r}{2}\right) = \emptyset$ *for* $i \neq j$. *Moreover every point of* X *is covered by at most* M *balls, where* M *depends only on* r.

We prove now a compact embedding result

Lemma 2. *Let the assumption related to inequality* (1.2) *holds. Then every sequence* $\{u_n\}$ *in* $\mathcal{C}\,[B(x,(k+1)R)]$ *such that*

$$\int\limits_{B(x,kr)} d\mathcal{L}[u] \leq C \tag{2.1}$$

is relatively compact in $L^p(B(x,R), m)$.

Proof. We have to prove that there is a subsequence of $\{u_n\}$ convergent in $L^p(B(x,R),m)$. Taking into account assumption (1.1), the ball $B(x,R)$ can be covered by a finite number of balls $B(x_i,r)$, $r \le \frac{R}{4}$, $j = 1, \dots, Q$ where Q depends on r, R, such that every point of $B(x,R)$ belongs at most to M balls, where M does not depend on r. Let $w_{n,m} = u_n - u_m$ and $\bar{w}_{n,m} = \int\limits_{B(x_j,r)} w_{n,m} m(dx)$. Then

$$\int\limits_{B(x,R)} w_{n,m}^p m(dx) \le \sum_{j=1}^{Q} \int\limits_{B(x_j,r)} w_{n,m}^p m(dx) = \sum_{j=1}^{Q} \int\limits_{B(x_j,r)} |w_{n,m} - \bar{w}_{n,m} + \bar{w}_{n,m}|^p m(dx)$$

$$\le 2^{p-1} \sum_{j=1}^{Q} \int\limits_{B(x_j,r)} |w_{n,m} - \bar{w}_{n,m}|^p m(dx) + 2^{p-1} \sum_{j=1}^{Q} \int\limits_{B(x_j,r)} (\bar{w}_{n,m})^p m(dx). \tag{2.2}$$

Since

$$\int\limits_{B(x_j,r)} (\bar{w}_{n,m})^p m(dx) = \int\limits_{B(x_j,r)} \frac{m(dx)}{m^p(B(x_j,r))} \left(\int\limits_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p$$

$$= \frac{1}{m^{p-1}(B(x_j,r))} \left(\int\limits_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p, \tag{2.3}$$

then inequality (2.2) becomes

$$2^{p-1} \sum_{j=1}^{Q} \int\limits_{B(x_j,r)} |w_{n,m} - \bar{w}_{n,m}|^p m(dx) + 2^{p-1} \sum_{j=1}^{Q} \int\limits_{B(x_j,r)} (\bar{w}_{n,m})^p m(dx)$$

$$\le 2^{p-1} c_p r^\alpha \sum_{j=1}^{Q} \int\limits_{B(x_j,kr)} d\mathcal{L}[u] + 2^{p-1} \sum_{j=1}^{Q} \frac{1}{m^{p-1}(B(x_j,r))} \left(\int\limits_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p$$

(2.4)

$$\le 2^{p-1} c_p r^\alpha M C k^\nu + \left(\frac{R}{r}\right)^{\nu(p-1)} \frac{2^{p-1}}{m^{p-1}(B(x,R)) c_0} \sum_{j=1}^{Q} \left(\int\limits_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p.$$

Choose $r = r_\varepsilon$ and $\varepsilon > 0$ such that $2^{p-1} c_p r_\varepsilon^\alpha M C k^\nu \le \frac{\varepsilon}{2}$. Suppose $\{u_n\}$ is weakly convergent in $L^p(B(x,(k+1)R),m)$ then

$$\left(\frac{R}{r_\varepsilon}\right)^{\nu(p-1)} \frac{2^{p-1}}{m^{p-1}(B(x,R)) c_0} \sum_{j=1}^{Q} \left(\int\limits_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p \le \frac{\varepsilon}{2} \tag{2.5}$$

for $n, m \geq n_\varepsilon$. This implies

$$\int\limits_{B(x,R)} w_{n,m}^p m\,(dx) \leq \varepsilon \tag{2.6}$$

and $\{u_n\}$is a Cauchy sequence in the space $L^p\,(B\,(x,R)\,,m)$ then $\{u_n\}$is convergent in $L^p\,(B\,(x,R)\,,m)$. □

Lemma 3. *Let $W \subset \mathcal{C}$ be the space defined in Eq.* (1.6) *and let us assume that W be a Banach space w.r.t. $\|.\|_W$, then the embedding of W in $L^p\,(X,m)$ is compact.*

Proof. Let $\|u_k\|_W \leq C$. After extraction of a subsequence, we have that $\{u_k\}$ is weakly convergent in W to u. We suppose, without loss of generality that $u = 0$ and prove

$$\int\limits_X u_k^p m\,(dx) \to 0 \tag{2.7}$$

when $k \to +\infty$. Let $\varepsilon > 0$, $\exists\ R > 0$ such that $V\,(x) \geq \frac{1+C^p}{\varepsilon}$ when $d\,(x,0) \geq R$. Since $\int\limits_{B(0,R)} u_k^p m\,(dx) \to 0$ when $k \to +\infty$, then $\exists\ k$ such that for $k \geq k_\varepsilon$

$$\int\limits_{B(0,R)} u_k^p m\,(dx) \leq \frac{\varepsilon}{1+C^p}. \tag{2.8}$$

Then for $k \geq k_\varepsilon$

$$\int\limits_X u_k^p m\,(dx) = \int\limits_{B(0,R)} u_k^p m\,(dx) + \int\limits_{X\setminus B(0,R)} u_k^p m\,(dx)$$

$$\leq \frac{\varepsilon}{1+C^p} + \int\limits_{X\setminus B(0,R)} u_k^p m\,(dx) \leq \frac{\varepsilon}{1+C^p}\left[1 + \int\limits_{X\setminus B(0,R)} V u_k^p m\,(dx)\right]$$

$$\leq \frac{\varepsilon}{1+C^p}\left[1 + \|u_k\|_W^p\right] \leq \varepsilon. \tag{2.9}$$

□

3. Proof of Theorem1

The function on W associated to our problem can be written as

$$\varphi\,(u) = \frac{1}{2}\,\|u\|_W^p - \int\limits_X F\,(u\,(x))\,m\,(dx)\,. \tag{3.1}$$

It can be proved that $\varphi \in C^1\,(W,\mathbb{R})$ and

$$\langle \varphi'\,(u)\,,v\rangle = (u,v)_W - \int\limits_X f\,(x,u\,(x))\,v\,(x)\,m\,(dx)\,. \tag{3.2}$$

The critical points of φ are weak solution of our problem, then to prove Theorem1 it is enough to prove the existence of nontrivial points for φ.

Proposition 1. *The functional φ satisfies the Palais-Smale condition under assumption of Theorem 1*

Proof. Let $\{u_k\}$ be a sequence in W such that

$$|\varphi(u_k)| \le C \qquad \varphi'(u_k) \to 0, \tag{3.3}$$

in W^* as $k \to +\infty$, where W^* denotes the dual space of W. From (3.3) we obtain that there exists k_0 such that for $k \ge k_0$

$$|\langle \varphi'(u_k), u_k\rangle| \le \mu \|u_k\|_W . \tag{3.4}$$

Then

$$C + \|u\|_W^p \ge \varphi(u_k) - \frac{1}{\mu}\langle \varphi'(u_k), u_k\rangle$$

$$= \frac{1}{2}\|u_k\|_W^p - \int_X F(u_k(x))\, m(dx) - \frac{1}{\mu}\left(\|u_k\|_W^p - \int_X f(u_k(x))\, u_k m(dx)\right)$$

(3.5)

$$= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_W^p - \int_X F(u_k(x))\, m(dx) - \frac{1}{\mu}\int_X f(u_k(x))\, u_k m(dx) \ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_W^p .$$

Since $\{u_k\}$ is bounded in W and from Lemma 3, we know that there exists a subsequence strongly convergent in $L^p(X,m)$ and weakly to $u \in W$. We apply now the Lemma 5 if $\nu \ge p$ or the Lemma 6 if $\nu < p$ of Ref.[4] to the function $g(t) = f(t)$ and to the sequence (u_k) and we obtain

$$\lim_{k\to+\infty} \int_X f(u_k)(u_k - u)\, m(dx) = 0. \tag{3.6}$$

From the assumption we have that

$$|\langle \varphi'(u), v\rangle| \le \varepsilon_k \|v\|_W \tag{3.7}$$

where $\varepsilon_k \to 0$ as $k \to +\infty$. Then from (3.7) we have

$$\langle \varphi'(u_k), u_k - u\rangle = (u_k, u_k - u)_W - \int_X f(x, u_k(x))(u_k - u)(x)\, m(dx)$$

$$= \|u_k\|_W^p - (u_k, u)_W - \int_X f(x, u_k(x))(u_k - u)(x)\, m(dx) . \tag{3.8}$$

From (3.6) and (3.7) we obtain

$$\langle \varphi'(u_k), u_k - u\rangle \to \|u_k\|_W^p - (u_k, u)_W \to 0, \tag{3.9}$$

when $k \to +\infty$. This implies that $\{u_k\}$ converges to u strongly in W. $\square$

Proof of Theorem1. First we prove that for $\rho \le \min\left(\frac{a}{2} m(B(0,1)), \frac{1}{2}\right)$ small enough $\varphi(u) \ge \gamma > 0$ for $\|u_k\|_W = \rho$. Consider the case $\nu \ge p$. As in Lemma 5 of Ref.[4] we obtain that for every $\varepsilon > 0$ there exists a constant C_ε such that

$$0 \le F(t) \le \varepsilon\left(|t|^p + |t|^\beta\right) + C_\varepsilon |t|^\beta \tag{3.10}$$

where $\beta = \frac{p\nu}{\nu - p}$ if $\nu > p$ or $\beta = \sigma + 1$ if $\nu = p$. There exists C such that

$$\|u\|_{L^p(X,m)} \le C\|u\|_W , \qquad \|u\|_{L^\beta(X,m)} \le C\|u\|_W . \tag{3.11}$$

Choose $\varepsilon < \frac{1}{2C^p}$; then

$$\int\limits_X F(u)\, m(dx) \le \varepsilon \left[\int\limits_X |u|^p\, m(dx) + \int\limits_X |u|^\beta\, m(dx) \right] + C_\varepsilon \int\limits_X |u|^\beta\, m(dx)$$

(3.12)

$$= \varepsilon \left(\|u\|^p_{L^p(X,m)} + \|u\|^\beta_{L^\beta(X,m)} \right) + C_\varepsilon \|u\|^\beta_{L^\beta(X,m)} \le \varepsilon \left(C^p \|u\|^p_W + C^\beta \|u\|^\beta_W \right) + C_\varepsilon C^\beta \|u\|^\beta_W$$

and

$$\varphi(u) = \frac{1}{2} \|u\|^p_W - \int\limits_X F(u)\, m(dx) \ge \left(\frac{1}{2} - \varepsilon C^p \right) \|u\|^p_W - C^\beta (\varepsilon + C_\varepsilon) \|u\|^\beta_W$$

$$\ge \rho^p - C^\beta (\varepsilon + C_\varepsilon) \rho^\beta \tag{3.13}$$

and the result follows from the last inequality. We consider now the case $\nu < 2$. From the assumption we obtain that for every $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$F(t) \le \varepsilon |t|^p \tag{3.14}$$

for $|t| \le \delta$. We observe that there exists C such that

$$\|u\|_{L^p(X,m)} \le C \|u\|_W, \qquad \|u\|_{L^\infty(X,m)} \le C \|u\|_W. \tag{3.15}$$

Choosing $\|u\|_W = \rho = \frac{\delta}{C}$, we have $\|u\|_{L^\infty(X,m)} \le \delta$; then

$$\int\limits_X F(u)\, m(dx) \le \varepsilon \int\limits_X |u|^p\, m(dx) = \varepsilon \|u\|^p_{L^p(X,m)} \le \varepsilon C^p \|u\|^p_W \tag{3.16}$$

and

$$\varphi(u) = \frac{1}{2} \|u\|^p_W - \int\limits_X F(u)\, m(dx) \ge \left(\frac{1}{2} - \varepsilon C^p \right) \|u\|^p_W \ge \rho^p. \tag{3.17}$$

The result follows from the last inequality. Let us prove the existence of $u_0 \in X \backslash B_\rho$ such that $\varphi(u) \le 0$. Let $u_0 \in D[a]$ be the potential of the ball $B(0,1)$ with respect to the ball $B(0,2)$. Then u_0 is in W and $\|u_0\|_W \ge am(B(0,1)) > \rho$; we recall that

$$F(u_0(x)) \ge m |u_0(x)|^\mu \tag{3.18}$$

for $x \in B(0,1)$. Let $\gamma > 1$; we have $u_0(x) = 1$ on $B(0,1)$, so

$$\varphi(\gamma u_0) = \frac{1}{2} \gamma^p \|u_0\|^p_W - \int\limits_X F(\gamma u_0)\, m(dx) \le \frac{1}{2} \gamma^p \|u_0\|^p_W - \int\limits_{B(0,1)} F(\gamma u_0)\, m(dx)$$

$$\le \frac{1}{2} \gamma^p \|u_0\|^p_W - m\gamma^\mu \int\limits_{B(0,1)} |u_0|^\mu\, m(dx) \le \frac{1}{2} \gamma^p \|u_0\|^p_W - m\gamma^\mu m(B(0,1)). \tag{3.19}$$

Since $\mu > p$ we have for $\gamma > \gamma_0$, γ_0 suitable, we have $\varphi(\gamma u_0) < 0$. The proof is completed with the application of the Mountain Pass Theorem. $\square$

References

[1] Fukushima M., *Dirichlet forms and Markov processes,* North Holland Math. Library (1980).
[2] Biroli M., Mosco U., Ann. Mat. Pura Appl. (IV), **169**, (1995), pp. 125-181.
[3] Malý J., Mosco U., Ric. Mat. **48**, Suppl. (1999), 217-231.
[4] Biroli M., Tersian S., Istituto Lombardo (Rend. Sc.) A **131**, 151-168 (1997).
[5] Falconer K.J., Commun. Math. Phys. **206**, (1999), 235-245.
[6] Matzeu, M., Appell, Juergen (ed.), Recent trends in nonlinear analysis. Festschrift dedicated to Alfonso Vignoli on the occasion of his 60th birthday. Basel: Birkhaeuser. Prog. Nonlinear Differ. Equ. Appl. 40, 217-231 (2000).